Gluing hyperconvex metric spaces

Benjamin Miesch

Abstract

We investigate how to glue hyperconvex (or injective) metric spaces such that the resulting space remains hyperconvex. We give two new criteria, saying that on the one hand gluing along strongly convex subsets and on the other hand gluing along externally hyperconvex subsets leads to hyperconvex spaces. Furthermore, we show by an example that these two cases where gluing works are opposed and cannot be combined.

1 Introduction

A metric space (X, d) is called *hyperconvex* if every collection of closed balls $\{B(x_i, r_i)\}_{i \in I}$ with $d(x_i, x_j) \leq r_i + r_j$ has non-empty intersection $\bigcap_i B(x_i, r_i) \neq \emptyset$. Hyperconvex spaces were introduced by Aronszajn and Panitchpakdi [1] who proved that they are the same as injective metric spaces. Hyperconvex spaces play a crucial role in metric fixed point theory, see [4] and the references therein.

A classical problem that arises in metric geometry is how to glue metric spaces such that their properties are preserved. Some attempts to solve this question for hyperconvex spaces can be found in [8], where it is shown that gluing along unique intervals preserves hyperconvexity.

As we will show, this result can be generalized to strongly convex subsets. A subset A of a metric space (X, d) is called *strongly convex* if for each pair $x, y \in A$ the metric interval $I(x, y) = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$ is contained in A. In section 3 we prove the following theorem.

Theorem 1.1. *Let (X, d) be the metric space obtained by gluing a collection $(X_\lambda, d_\lambda)_{\lambda \in \Lambda}$ of hyperconvex metric spaces along some space A such that A is closed and strongly convex in X_λ for each $\lambda \in \Lambda$. Then (X, d) is hyperconvex as well.*

Interesting objects that can be obtained by gluing are polyhedral or cubical complexes. One attempt to put a hyperconvex metric on cube complexes is to take a metric on each cube such that they are isometric to the unit cube $[0, 1]^n$ in l_∞^n (see [5, 6]). As we see there the gluing of two cubes along some face preserves hyperconvexity. But for $n \geq 2$ proper faces of n-cubes are far from being strongly convex. The aim of our second criterion is to characterize such sets in arbitrary hyperconvex spaces and show that gluing along these subsets preserves hyperconvexity in general.

A subset A of a metric space X is called *externally hyperconvex* (cf. [1]) if for any collection of balls $\{B(x_i, r_i)\}_{i\in I}$ in X with $d(x_i, x_j) \leq r_i + r_j$ and $d(x_i, A) \leq r_i$ we have $A \cap \bigcap_i B(x_i, r_i) \neq \emptyset$. In section 4 we first show that bounded externally hyperconvex subsets share some basic properties of balls in hyperconvex spaces.

Proposition 1.2. *Let (X, d) be a hyperconvex space and $\{A_i\}_{i\in I}$ a family of pairwise intersecting externally hyperconvex subsets such that one of them is bounded. Then $\bigcap_{i\in I} A_i \neq \emptyset$.*

Using this crucial property we prove that gluing along externally hyperconvex subsets preserves hyperconvexity. This answers a question raised in [8].

Theorem 1.3. *Let (X, d) be the metric space obtained by gluing a family of hyperconvex metric spaces $(X_\lambda, d_\lambda)_{\lambda\in\Lambda}$ along some set A such that A is externally hyperconvex in each X_λ. Then X is hyperconvex. Moreover A is externally hyperconvex in X.*

Finally we give an example showing that gluing two spaces X_1, X_2 along some subset A such that $A \subset X_1$ is strongly convex and $A \subset X_2$ is externally hyperconvex does not preserve hyperconvexity in general.

2 Preliminaries

First we fix some notation. Let (X, d) be a metric space. We denote by

$$B(x_0, r) = \{x \in X : d(x, x_0) \leq r\}$$

the closed ball of radius r with center in x_0. For any subset $A \subset X$ let

$$B(A, r) = \{x \in X : d(x, A) := \inf_{y\in A} d(x, y) \leq r\}$$

be the closed r-neighborhood of A.

Definition 2.1. Let $(X_\lambda, d_\lambda)_{\lambda\in\Lambda}$ be a family of metric spaces with closed subspaces $A_\lambda \subset X_\lambda$. Suppose that all A_λ are isometric to some metric space A. For every $\lambda \in \Lambda$ fix some isometry $\varphi_\lambda\colon A \to A_\lambda$. We define an equivalence relation on the disjoint union $\bigsqcup_\lambda X_\lambda$ generated by $\varphi_\lambda(a) \sim \varphi_{\lambda'}(a)$ for $a \in A$. The resulting space $X = (\bigsqcup_\lambda X_\lambda)/\sim$ is called the *gluing* of the X_λ along A.

X admits a natural metric. For $x \in X_\lambda$ and $y \in X_{\lambda'}$ it is given by

$$d(x, y) = \begin{cases} d_\lambda(x, y), & \text{if } \lambda = \lambda', \\ \inf_{a\in A}\{d_\lambda(x, \varphi_\lambda(a)) + d_{\lambda'}(\varphi_{\lambda'}(a), y)\}, & \text{if } \lambda \neq \lambda'. \end{cases} \tag{1}$$

For details see for instance Lemma I.5.24 in [3].

Hereinafter, if there is no ambiguity, indices for d_λ are dropped and the sets $A_\lambda = \varphi_\lambda(A) \subset X_\lambda$ are identified with A.

Definition 2.2. A metric space (X, d) is *injective* if for every isometric embedding $\iota\colon A \hookrightarrow Y$ of metric spaces and every 1-lipschitz map $f\colon A \to X$ there is some 1-lipschitz map $\bar{f}\colon Y \to X$ such that $f = \bar{f} \circ \iota$.

Injective metric spaces are complete, geodesic and contractible. Moreover they have the following intersection property.

Definition 2.3. A metric space (X, d) is *(m-)hyperconvex* if for any collection $\{B(x_i, r_i)\}_{i \in J}$ of closed balls with $d(x_i, x_j) \leq r_i + r_j$ (and $|J| \leq m$) we have

$$\bigcap_{i \in J} B(x_i, r_i) \neq \emptyset.$$

Proposition 2.4. *(Theorem 4 in [1]). A metric space (X, d) is injective if and only if it is hyperconvex.*

Lemma 2.5. *Let (X, d) be a (3-)hyperconvex metric space. For $x, y, z \in X$ we have*

$$I(x, y) \cap I(y, z) \cap I(z, x) \neq \emptyset.$$

Proof. Choose $\alpha, \beta, \gamma \geq 0$ such that

$$\begin{aligned} \alpha + \beta &= d(x, y), \\ \alpha + \gamma &= d(x, z), \\ \beta + \gamma &= d(y, z). \end{aligned}$$

Then $I(x, y) \cap I(y, z) \cap I(z, x) = B(x, \alpha) \cap B(y, \beta) \cap B(z, \gamma) \neq \emptyset$. $\square$

Definition 2.6. A subset A of a metric space (X, d) is *proximinal* if for all $x \in X$ the intersection $B(x, d(x, A)) \cap A$ is non-empty.

Remark. Proximinal subsets are closed.

3 Gluing along strongly convex subsets

Definition 3.1. A subset A of a metric space (X, d) is *gated* if for all $x \in X$ there is some $\bar{x} \in A$ such that for all $a \in A$ we have $d(x, a) = d(x, \bar{x}) + d(\bar{x}, a)$. Clearly if such an $\bar{x}$ exists it is unique and we then call $\bar{x}$ the *gate* of x in A.

Lemma 3.2. (cf. Lemma 1.82 in [7]). *Let A be a subset of the (3-)hyperconvex metric space (X, d). Then A is strongly convex and closed if and only if it is gated.*

Proof. First assume that A is strongly convex and closed. Fix $x \in X$. Let x_n be a sequence of points in A with $d(x, x_n) \leq d(x, A) + \frac{1}{n}$. For $n, k \in \mathbb{N}$ take $m_{n,k} \in I(x, x_n) \cap I(x, x_k) \cap I(x_n, x_k)$. By strong convexity we get $m_{n,k} \in A$ and hence

$$d(x_n, m_{n,k}) = d(x, x_n) - d(x_n, m_{n,k}) \leq d(x, A) + \frac{1}{n} - d(x, A) = \frac{1}{n}.$$

By interchanging x_n and x_k we also get $d(x_k, m_{n,k}) \leq \frac{1}{k}$. Therefore

$$d(x_n, x_k) = d(x_n, m_{n,k}) + d(m_{n,k}, x_k) \leq \frac{1}{n} + \frac{1}{k},$$

i.e. x_n is a Cauchy sequence and since A is closed it converges to some $\bar{x} \in A$. Moreover we have $d(x, \bar{x}) = d(x, A)$. We claim that $\bar{x}$ is a gate for x in A. Let $y \in A$. By Lemma 2.5 there is some $z \in I(x, \bar{x}) \cap I(\bar{x}, y) \cap I(y, x)$. By convexity we have $z \in A$ and therefore $d(x, z) \geq d(x, A) = d(x, \bar{x})$. Since $z \in I(x, \bar{x})$ this implies $z = \bar{x}$ and hence $\bar{x} \in I(x, y)$ as desired. On the other hand, if A is gated for all points $x, y \in A$ and $z \in I(x, y)$ we have $z = \bar{z}$ and hence $I(x, y) \subset A$. Moreover for all $x \in X$ we have $d(x, A) = d(x, \bar{x})$ and therefore $\bar{x} \in B(x, d(x, A)) \cap A$, i.e. A is proximinal and therefore closed. □

Lemma 3.3. *Let A be a gated subset of a (m-)hyperconvex metric space (X, d). Then (A, d) is (m-)hyperconvex as well.*

Proof. Let $\{B(x_i, r_i)\}_{i \in J}$ be a collection of closed balls in X with $d(x_i, x_j) \leq r_i + r_j$ and centers in A. Since X is hyperconvex there is some $z \in \bigcap_{i \in J} B(x_i, r_i)$. Let $\bar{z}$ be the gate of z in A. Then $\bar{z} \in \bigcap_{i \in J} B(x_i, r_i) \cap A$ since $d(x_i, \bar{z}) = d(x_i, z) - d(z, \bar{z}) \leq r_i$ for all $i \in J$. □

Lemma 3.4. *Let X be a metric space obtained by gluing the metric spaces $(X_\lambda, d_\lambda)_{\lambda \in \Lambda}$ along some set A. Assume that A is gated in X_λ for all λ. Then for $x \in X_\lambda, y \in X_{\lambda'}$ with $\lambda \neq \lambda'$ we have*

$$d(x, y) = d(x, \bar{x}) + d(\bar{x}, \bar{y}) + d(\bar{y}, y). \tag{2}$$

Proof. For $a \in A$ we have

$$\begin{aligned} d(x, a) + d(a, y) &= d(x, \bar{x}) + d(\bar{x}, a) + d(a, \bar{y}) + d(\bar{y}, y) \\ &\geq d(x, \bar{x}) + d(\bar{x}, \bar{y}) + d(\bar{y}, y). \end{aligned}$$

□

Proposition 3.5. *Let (X, d) be the metric space obtained by gluing the collection $(X_\lambda, d_\lambda)_{\lambda \in \Lambda}$ of (m-)hyperconvex metric spaces along some space A such that A is closed and strongly convex in all X_λ. Then (X, d) is (m-)hyperconvex as well.*

Proof. Let $\{B(x_i, r_i)\}_{i \in I}$ be a collection of closed balls in X with $d(x_i, x_j) \leq r_i + r_j$. We need to show that $\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$. For any point $x \in X$ denote by $\bar{x}$ its gate in A. Moreover if $d(x_i, \bar{x}_i) \leq r_i$ define $\bar{r}_i = r_i - d(x_i, \bar{x}_i)$. Observe that $B(\bar{x}_i, \bar{r}_i) \subset B(x_i, r_i)$. We distinguish three cases.
Case 1. First we assume $d(x_i, \bar{x}_i) \leq r_i$ and $B(\bar{x}_i, \bar{r}_i) \cap B(\bar{x}_j, \bar{r}_j) \neq \emptyset$ for all $i, j \in I$.

Since A is hyperconvex we have $\bigcap_{i \in I} B(\bar{x}_i, \bar{r}_i) \neq \emptyset$ and hence

$$\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset.$$

Case 2. Let again be $d(x_i,\bar{x}_i) \le r_i$ for all $i \in I$ but assume that there are $i,j \in I$ such that

$$B(\bar{x}_i,\bar{r}_i) \cap B(\bar{x}_j,\bar{r}_j) = \emptyset, \tag{3}$$

i.e. $d(x_i,\bar{x}_i) + d(\bar{x}_i,\bar{x}_j) + d(\bar{x}_j,x_j) > r_i + r_j$.

Observe that by Lemma 3.4 we have $B(\bar{x}_i,\bar{r}_i) \cap B(\bar{x}_j,\bar{r}_j) \neq \emptyset$ if $x \in X_\lambda$, $y \in X_{\lambda'}$ with $\lambda \neq \lambda'$ and therefore any two x_i, x_j fulfilling (3) must be contained in some X_λ. We now claim that there is only one X_{λ_0} containing such pairs.

Let $x_1, x_2 \in X_{\lambda_0}$ be such that

$$B(\bar{x}_1,\bar{r}_1) \cap B(\bar{x}_2,\bar{r}_2) = \emptyset.$$

and $x_3, x_4 \in X_\lambda$ for some $\lambda \neq \lambda_0$. Define $r := \frac{d(\bar{x}_1,\bar{x}_2)-\bar{r}_1-\bar{r}_2}{2}$. We have

$$B(\bar{x}_1,\bar{r}_1 + r) \cap B(\bar{x}_2,\bar{r}_2 + r) \neq \emptyset$$

and therefore since X_λ is hyperconvex there is some

$$z \in B(\bar{x}_1,\bar{r}_1 + r) \cap B(\bar{x}_2,\bar{r}_2 + r) \cap B(x_3,r_3) \cap B(x_4,r_4).$$

But by the choice of r we have $z \in I(x_1,x_2) \subset A$ and thus for $i = 3,4$ we get

$$d(\bar{x}_i,z) = d(x_i,z) - d(\bar{x}_i,x_i) \le r_i - d(\bar{x}_i,x_i) = \bar{r}_i.$$

Hence we conclude $z \in B(\bar{x}_3,\bar{r}_3) \cap B(\bar{x}_4,\bar{r}_4) \neq \emptyset$.

Denote $I_0 = \{i \in I : x_i \in X_{\lambda_0}\}$. Then $\{B(x_i,r_i)\}_{i\in I_0} \cup \{B(\bar{x}_i,\bar{r}_i)\}_{i\in I\setminus I_0}$ is a family of pairwise intersecting balls in X_{λ_0} and therefore has non-empty intersection $\bigcap_{i\in I_0} B(x_i,r_i) \cap \bigcap_{i\in I\setminus I_0} B(\bar{x}_i,\bar{r}_i)$ what implies $\bigcap_{i\in I} B(x_i,r_i) \neq \emptyset$.

Case 3. It remains the situation where $d(x_i,\bar{x}_i) > r_i$ for some $i \in J$.

From Lemma 3.4 it immediately follows that all such x_i are contained in some X_{λ_0}. Fix some $x_{i_0} \in X_{\lambda_0}$ with $d(x_{i_0},\bar{x}_{i_0}) > r_{i_0}$. Then for $x_i \notin X_{\lambda_0}$ we have $\bar{x}_{i_0} \in B(\bar{x}_i,\bar{r}_i)$ and therefore $B(\bar{x}_i,\bar{r}_i) \cap B(\bar{x}_j,\bar{r}_j) \neq \emptyset$ for $x_i, x_j \notin X_{\lambda_0}$. We conclude as above that $\bigcap_{i\in I} B(x_i,r_i) \neq \emptyset$. $\square$

4 Gluing along externally hyperconvex subspaces

In the following let (X,d) be a metric space.

Definition 4.1. A subset $A \subset X$ is called *admissible* if it can be written as an intersection of closed balls $A = \bigcap_{i\in I} B(x_i,r_i)$. We denote the family of all admissible subsets of X by $\mathcal{A}(X)$.

Remark. A family of pairwise intersecting admissible sets in a hyperconvex space has non-empty intersection.

Definition 4.2. A subset $A \subset X$ is called *externally hyperconvex* if for all collections $\{B(x_i,r_i)\}_{i\in I}$ with $d(x_i,x_j) \le r_i + r_j$ and $d(x_i,A) \le r_i$ we have $\bigcap_i B(x_i,r_i) \cap A \neq \emptyset$. Denote the set of externally hyperconvex subsets of X by $\mathcal{E}(X)$.

Remark. Externally hyperconvex subsets are proximinal and therefore closed.

First we give the proof of two well known facts for externally hyperconvex subsets.

Lemma 4.3. *If $A \in \mathcal{A}(X)$ and $E \in \mathcal{E}(X)$ such that $A \cap E \neq \emptyset$ then $A \cap E \in \mathcal{E}(X)$. Especially if X is hyperconvex we have $\mathcal{A}(X) \subset \mathcal{E}(X)$.*

Proof. Since A is admissible there is a collection of balls $\{B(x_i, r_i)\}_{i \in I}$ such that $A = \bigcap_i B(x_i, r_i)$. Now given a family of closed balls $\{B(x'_j, r'_j)\}_{j \in J}$ with $d(x'_j, x'_k) \leq r'_j + r'_k$ and $d(x'_j, A \cap E) \leq r'_j$ we have $d(x_i, x'_j) \leq r_i + r'_j$ and $d(x_i, E) \leq r_i$ and therefore

$$A \cap E \cap \bigcap_j B(x'_j, r'_j) = E \cap \bigcap_i B(x_i, r_i) \cap \bigcap_j B(x'_j, r'_j) \neq \emptyset$$

since E is externally hyperconvex. If X is hyperconvex then $X \in \mathcal{E}(X)$ and therefore $\mathcal{A}(X) \subset \mathcal{E}(X)$. □

Lemma 4.4. *Let X be hyperconvex and $A \in \mathcal{E}(X)$. Then also $B(A, r) \in \mathcal{E}(X)$.*

Proof. Let $\{B(x_i, r_i)\}_{i \in I}$ be a collection of closed balls with $d(x_i, x_j) \leq r_i + r_j$ and $d(x_i, B(A, r)) \leq r_i$. Then we also have $d(x_i, A) \leq r_i + r$. Since A is externally hyperconvex there is some $y \in \bigcap B(x_i, r_i + r) \cap A$. Especially we have $d(x_i, y) \leq r_i + r$ and therefore since X is hyperconvex we get $\emptyset \neq \bigcap B(x_i, r_i) \cap B(y, r) \subset \bigcap B(x_i, r_i) \cap B(A, r)$. □

The following technical lemma turns out to be the initial step in proving Proposition 1.2.

Lemma 4.5. *Let X be a hyperconvex space. Let $A, A' \in \mathcal{E}(X)$ with $y \in A \cap A' \neq \emptyset$ and $x \in X$ with $d(x, A), d(x, A') \leq r$. Denote $d := d(x, y)$ and $s := d - r$. Then $A \cap A' \cap B(x, r) \cap B(y, s) \neq \emptyset$, given $s \geq 0$. In any case the intersection $A \cap A' \cap B(x, r)$ is non-empty.*

Proof. For $s \leq 0$ we have $y \in A \cap A' \cap B(x, r)$ Therefore let us assume $s > 0$.

Claim. For each $0 < l \leq s$ there are $a \in A, a' \in A'$ such that $d(a, a') \leq l$ and $a, a' \in B(x, r) \cap B(y, s)$.

We start choosing

$$a_1 \in B(y, l) \cap B(x, d - l) \cap A$$

and

$$a'_1 \in B(y, l) \cap B(x, d - l) \cap B(a_1, l) \cap A'.$$

Then we inductively take

$$a_n \in B(y, nl) \cap B(x, d - nl) \cap B(a'_{n-1}, l) \cap A$$

and

$$a'_n \in B(y, nl) \cap B(x, d - nl) \cap B(a_n, l) \cap A'$$

as long as $n \leq \lfloor \frac{s}{l} \rfloor =: n_0$. Finally there are

$$a \in B(y,s) \cap B(x,r) \cap B(a'_{n_0}, l) \cap A$$

and

$$a' \in B(y,s) \cap B(x,r) \cap B(a,l) \cap A'$$

as desired.

We now construct recursively two converging sequences $(a_n)_n \subset A$ and $(a'_n)_n \subset A'$ such that $a_n, a'_n \in B(x,r) \cap B(y,s)$ with

$$d(a_n, a'_n) \leq \frac{1}{2^{n+1}} \text{ and } d(a_{n-1}, a_n), d(a'_{n-1}, a'_n) \leq \frac{1}{2^n}.$$

First choose $a_0, a'_0 \in B(x,r) \cap B(y,s)$ with $d(a_0, a'_0) \leq \frac{1}{2}$ according to the claim. Given a_{n-1}, a'_{n-1} with $d(a_{n-1}, a'_{n-1}) \leq \frac{1}{2^n}$ there is some $x_n \in B(a_{n-1}, \frac{1}{2^{n+1}}) \cap B(a'_{n-1}, \frac{1}{2^{n+1}}) \cap B(x, r - \frac{1}{2^{n+1}})$. Now applying the claim to x_n and y we find

$$a_n, a'_n \in B(y,s) \cap B(x_n, \tfrac{1}{2^{n+1}}) \subset B(y,s) \cap B(x,r)$$

with $d(a_n, a'_n) \leq \frac{1}{2^{n+1}}$. Moreover we have

$$d(a_{n-1}, a_n) \leq d(a_{n-1}, x_n) + d(x_n, a_n) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}.$$

For $m \geq n$ we get

$$d(a_n, a_m) \leq \sum_{k=n+1}^{m} d(a_{k-1}, a_k) \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}$$

and similarly for a'_n. Hence the two sequences converge and since $d(a_n, a'_n) \to 0$ they have a common limit point $a \in B(y,s) \cap B(x,r) \cap A \cap A'$. $\square$

Lemma 4.6. *Let X be a hyperconvex space and let $A_0, A_1, A_2 \in \mathcal{E}(X)$ be pairwise intersecting externally hyperconvex subspaces. Then $A_0 \cap A_1 \cap A_2 \neq \emptyset$.*

Proof. Choose some point $x_0 \in A_1 \cap A_2$ and let $r := d(x_0, A_0)$. By the previous lemma there is $y_0 \in A_0 \cap A_1 \cap B(x_0, r)$. Define $A'_0 := A_0 \cap B(y_0, r) \in \mathcal{E}(X)$. Using again the lemma we have $A'_0 \cap A_2 = A_0 \cap A_2 \cap B(y_0, r) \neq \emptyset$ and therefore there is some $z_0 \in A'_0 \cap A_2 \cap B(x_0, r) = A_0 \cap A_2 \cap B(x_0, r) \cap B(y_0, r)$. Then since A_0 is externally hyperconvex, there is some $\bar{x}_0 \in B(x_0, r) \cap B(y_0, \frac{r}{2}) \cap B(z_0, \frac{r}{2}) \cap A_0$ and using again Lemma 4.5, we find $x_1 \in A_1 \cap A_2 \cap B(\bar{x}_0, \frac{r}{2}) \cap B(x_0, \frac{r}{2})$. Proceeding this way we get some sequence $(x_n)_n \subset A_1 \cap A_2$ with $d(x_n, A_0) \leq \frac{r}{2^n}$ and $d(x_{n-1}, x_n) \leq \frac{r}{2^n}$. Hence $d(x_n, x_m) \leq \sum_{k=n+1}^{m} \frac{r}{2^k} \leq \frac{r}{2^n}$ and therefore $(x_n)_n$ converges to $x \in A_0 \cap A_1 \cap A_2$. $\square$

Lemma 4.7. *If $A_0, A_1 \in \mathcal{E}(X)$ and $A_0 \cap A_1 \neq \emptyset$ then $A_0 \cap A_1 \in \mathcal{E}(X)$.*

Proof. Let $B(x_i, r_i)$ be a collection of balls with $d(x_i, x_j) \leq r_i + r_j$ and $d(x_i, A_1 \cap A_2) \leq r_i$. Define $A := \bigcap B(x_i, r_i)$. Since the A_k are externally hyperconvex we have $A \cap A_k = \bigcap_i B(x_i, r_i) \cap A_k \neq \emptyset$ and since admissible sets are externally hyperconvex we have $A_0 \cap A_1 \cap A \neq \emptyset$ by Lemma 4.6. $\square$

By induction we therefore get the following proposition.

Proposition 4.8. *Let $A_0, \dots, A_n \in \mathcal{E}(X)$ with $A_i \cap A_j \neq \emptyset$ then $\emptyset \neq \bigcap_{k=0}^n A_k \in \mathcal{E}(X)$.*

As a consequence of Baillon's theorem on the intersection of hyperconvex spaces [2] the following theorem was proven by Espínola and Khamsi in [4].

Theorem 4.9. *[4, Theorem 5.4]. Let $\{A_i\}_{i\in I}$ be a descending chain of non-empty externally hyperconvex subsets of a bounded hyperconvex space X. Then $\bigcap_i A_i$ is non-empty and externally hyperconvex in X.*

Similarly to Corollary 8 in [2] we can deduce the following corollary which implies Proposition 1.2.

Corollary 4.10. *Let $\{A_i\}_{i\in I}$ be a family of pairwise intersecting externally hyperconvex subsets of a bounded hyperconvex space X. Then $\bigcap_i A_i$ is non-empty and externally hyperconvex in X.*

Proof. Consider the set

$$\mathcal{F} = \left\{ J \subset I : \forall F \subset I \text{ finite}, \bigcap_{i \in J \cup F} A_i \neq \emptyset \text{ is externally hyperconvex} \right\}.$$

By Proposition 4.8 clearly $\emptyset \in \mathcal{F}$. Considering a chain $J_k \in \mathcal{F}$ and some finite set $F \subset I$, the sets $A_{J_k} = \bigcap_{i \in J_k \cup F} A_i$ build a decreasing chain of non-empty externally hyperconvex sets. Define $J = \bigcup_k J_k$. We have $A = \bigcap_{i\in J\cup F} A_i = \bigcap_k A_{J_k}$ is non-empty and externally hyperconvex by Theorem 4.9. Therefore $J \in \mathcal{F}$ is an upper bound of J_k. Hence $\mathcal{F}$ satisfies the hypothesis of Zorn's Lemma and therefore there is some maximal element $J_0 \in \mathcal{F}$. But for $i \in I$ we have $J_0 \cup \{i\} \in \mathcal{F}$ and by maximality of J_0 we conclude $I = J_0 \in \mathcal{F}$. □

Proposition 4.11. *Let Y be an externally hyperconvex subset of the metric space X. Moreover let A be externally hyperconvex in Y. Then A is also externally hyperconvex in X.*

Proof. Let $\{B(x_i, r_i)\}_{i\in I}$ be a collection of closed balls with $d(x_i, x_j) \leq r_i + r_j$ and $d(x_i, A) \leq r_i$. Then the sets $A_i := B(x_i, r_i) \cap Y$ are externally hyperconvex subsets of X and therefore also of Y. Clearly $A_i \cap A \neq \emptyset$ and since Y is externally hyperconvex we have $A_i \cap A_j = B(x_i, r_i) \cap B(x_j, r_j) \cap Y \neq \emptyset$. Therefore we have a collection of pairwise intersecting externally hyperconvex subsets of Y and hence by Proposition 1.2 $A \cap \bigcap_i B(x_i, r_i) = A \cap \bigcap_i A_i \neq \emptyset$. □

Before proving Theorem 1.3 we need some last technical lemmas. We use the convention that $B^\lambda(x, r)$ denotes the closed ball inside X_λ.

Lemma 4.12. *Let X be a metric space obtained by gluing the hyperconvex spaces X_λ along some set A with $A \in \mathcal{E}(X_\lambda)$. Let $x \in X_\lambda$ and $y \in X_{\lambda'}$ with $\lambda \neq \lambda'$. Then for $s = d(x, A)$ there is some $a \in A \cap B^\lambda(x, s)$ such that*

$$d(x, y) = d(x, a) + d(a, y).$$

Proof. Define $A' = A \cap B^\lambda(x,s) \neq \emptyset$. Observe that for $a \in A$ there is some $a' \in B^\lambda(x,s) \cap B^\lambda(a, d(a,x)-s) \cap A$ and hence

$$d(x,a) + d(a,y) = d(x,a') + d(a',a) + d(a,y) \geq d(x,a') + d(a',y).$$

Therefore

$$d(x,y) = \inf_{a \in A'} d(x,a) + d(a,y) = s + d(A',y) \tag{4}$$

By Lemma 4.3 and Proposition 4.11 we have $A' \in \mathcal{E}(X_{\lambda'})$. Thus there is some $a \in A' \cap B^{\lambda'}(y, d(A',y))$ and we get $d(x,y) = d(x,a) + d(a,y)$. □

Lemma 4.13. *Let X be a metric space obtained by gluing the hyperconvex spaces X_λ along some set A with $A \in \mathcal{E}(X_\lambda)$. Let $x \in X_\lambda$ and $r \geq s := d(x,A)$. Then for $\lambda_0 \neq \lambda$ we have*

$$B(x,r) \cap X_{\lambda_0} = B^{\lambda_0}(B^\lambda(x,s) \cap A, r-s). \tag{5}$$

Moreover $B(x,r) \cap X_{\lambda_0} \in \mathcal{E}(X_{\lambda_0})$.

Proof. Clearly $B(x,r) \cap X_{\lambda_0} \supset B^{\lambda_0}(B^\lambda(x,s) \cap A, r-s)$. Therefore assume $y \in B(x,r) \cap X_{\lambda_0}$. Then there is some $a \in A \cap B^\lambda(x,s)$ with $d(x,y) = d(x,a) + d(a,y)$ and $d(a,y) \leq r - s$ by Lemma 4.12.

We have $B^\lambda(x,s) \cap A \in \mathcal{E}(A)$ by Lemma 4.3 and therefore since $A \in \mathcal{E}(X_{\lambda_0})$ we also get $B^\lambda(x,s) \cap A \in \mathcal{E}(X_{\lambda_0})$ by Proposition 4.11. Finally we conclude by Lemma 4.4 that $B^{\lambda_0}(B^\lambda(x,s) \cap A, r-s) \in \mathcal{E}(X_{\lambda_0})$. □

Proof of Theorem 1.3. Let $\{B(x_i,r_i)\}_{i \in I}$ be a collection of closed balls in X with $d(x_i,x_j) \leq r_i + r_j$. First observe that there is at most one $\lambda_0 \in \Lambda$ such that $d(x_i,A) > r_i$ for some $x_i \in X_{\lambda_0}$. If there is none, fix any $\lambda_0 \in \Lambda$. Now define $A_i = B(x_i,r_i) \cap X_{\lambda_0} \neq \emptyset$. We claim that $A_i \cap A_j \neq \emptyset$ for all $i,j \in I$.

Let $x_i \in X_\lambda$ and $x_j \in X_{\lambda'}$. First assume $\lambda, \lambda' \neq \lambda_0$. If $\lambda = \lambda'$ we have $A_i \cap A_j \neq \emptyset$ since $A \in \mathcal{E}(X_\lambda)$. If $\lambda \neq \lambda'$ there is some $a \in A \cap B(x_i, d(x_i,A))$ with $d(x_i,x_j) = d(x_i,a) + d(a,x_j)$ and therefore $\emptyset \neq B^{\lambda'}(a, r_i - d(x_i,A)) \cap B^{\lambda'}(x_j,r_j) \cap A \subset A_i \cap A_j$ by external hyperconvexity of A in $X_{\lambda'}$. Finally if $\lambda' = \lambda_0$ we either get $B^{\lambda_0}(x_i,r_i) \cap B^{\lambda_0}(x_j,r_j) \neq \emptyset$ if $\lambda = \lambda'$ or $\emptyset \neq B^{\lambda_0}(a, r_i - d(x_i,A)) \cap B^{\lambda_0}(x_j,r_j) \subset A_i \cap A_j$ if $\lambda \neq \lambda'$ by hyperconvexity of X_{λ_0}.

Therefore using Lemma 4.13 the sets A_i are a collection of bounded pairwise intersecting externally hyperconvex subsets of X_{λ_0} and therefore have nonempty intersection $\bigcap_i A_i \neq \emptyset$ by Proposition 1.2, implying $\bigcap_i B(x_i,r_i) \neq \emptyset$.

To see that A is externally hyperconvex in X proceed as before adding A to the family $\{A_i\}_i$. □

5 Basic example

In l^2_∞ any halfspace is a hyperconvex subspace, where its boundary line is isometric to $\mathbb{R}$. It is natural to glue two such halfspaces along its boundaries. Up to isometry a halfspaces is given by $H = \{(\xi_1,\xi_2) \in l^2_\infty : \xi_2 \geq a\xi_1\}$ for $0 \leq a \leq 1$. Observe that the gluing set $A = \{(\xi_1,\xi_2) \in l^2_\infty : \xi_2 = a\xi_1\}$ is externally hyperconvex in H for $a = 0$ and strongly convex in H for $a = 1$.

Now given two halfspaces we may assume that $H_1 = \{(\xi_1, \xi_2) \in l_\infty^2 : \xi_2 \geq a\xi_1\}$ and $H_2 = \{(\xi_1, \xi_2) \in l_\infty^2 : \xi_2 \leq b\xi_1\}$ with $0 \leq a \leq b \leq 1$. Then there are two possibilities to glue them depending on the orientation of the boundary line.

Let us first investigate the case where we glue by identifying $(\xi, a\xi) \in H_1$ with $(\xi, b\xi) \in H_2$. Then the resulting space is hyperconvex if and only if $a = b$.

For $a = b$ we obtain a space which is isometric to l_∞^2 and therefore hyperconvex. Otherwise consider the pairwise intersecting closed balls of radius 1 and centers $x_1 = (0, 1-a) \in H_1$, $x_2 = (0, -b-1) \in H_2$ and $x_3 = (2, b-1) \in H_2$. The intersection of $B(x_1, 1)$ with H_2 is given by the union of all balls with center $y \in A \cap B(x_1, 1)$ and radius $r_y = 1 - d(x_1, y)$. A calculation shows that

$$B(x_1, 1) \cap H_2 = \left\{(\xi_1, \xi_2) \in H_2 : -1 \leq \xi_1 \leq 1 \text{ and } \xi_2 \geq \frac{b-a}{1+a}\xi_1 - \frac{1+b}{1+a}a\right\}$$

and therefore the three balls have no common intersection point for $a < b$.

It remains to consider the case where we reflect H_1 before gluing, i.e. $H_1 = \{(\xi_1, \xi_2) \in l_\infty^2 : \xi_2 \geq -a\xi_1\}$ and $H_2 = \{(\xi_1, \xi_2) \in l_\infty^2 : \xi_2 \leq b\xi_1\}$ with $0 \leq a \leq b \leq 1$ glued by identifying $(\xi, -a\xi) \in H_1$ with $(\xi, b\xi) \in H_2$. If $a = b = 1$ or $a = b = 0$ we have again that the resulting space is injective according to Theorems 1.1 and 1.3. Indeed in both cases it is isometric to l_∞^2. Otherwise choose again $x_1 = (0, 1-a) \in H_1$ as above. Then we get

$$B(x_1, 1) \cap H_2 = \{(\xi_1, \xi_2) \in H_2 : -1 \leq \xi_1 \leq 1 \text{ and } \xi_2 \geq \max\{l, m\xi_1 - q\}\}.$$

for $l = -1 + (1-b) \cdot \frac{1-a}{1+a} \leq -b$, $m = \frac{a+b}{1-a}$ and $q = a \cdot \frac{1+b}{1-a}$. Especially we have $(-1, l) \in B(x_1, 1) \cap H_2$ and $(1, \xi_2) \in B(x_1, 1) \cap H_2$ if and only if $\xi_2 = b$. Therefore if $a \neq 1$ and $b \neq 0$ the three balls with radius 1 and centers $x_1 = (0, 1-a) \in H_1$, $x_2 = (0, l-1) \in H_2$ and $x_3 = (2, b-1) \in H_2$ are pairwise intersecting but have no common intersection point. Hence the resulting space is not hyperconvex.

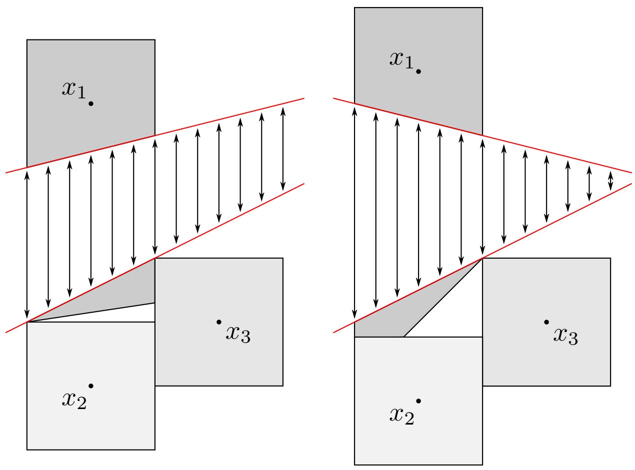

Figure 1: Gluing halfspaces in l_∞^2.

References

[1] N. Aronszajn and P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405–439. MR 0084762 (18,917c)

[2] Jean-Bernard Baillon, *Nonexpansive mapping and hyperconvex spaces*, Fixed point theory and its applications (Berkeley, CA, 1986), Contemp. Math., vol. 72, Amer. Math. Soc., Providence, RI, 1988, pp. 11–19. MR 956475 (89k:54068)

[3] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486 (2000k:53038)

[4] R. Espínola and M. A. Khamsi, *Introduction to hyperconvex spaces*, Handbook of metric fixed point theory, Kluwer Acad. Publ., Dordrecht, 2001, pp. 391–435. MR 1904284 (2003g:47099)

[5] Jie Hua Mai and Yun Tang, *An injective metrization for collapsible polyhedra*, Proc. Amer. Math. Soc. **88** (1983), no. 2, 333–337. MR 695270 (84g:54036)

[6] B. Miesch, *Injective Metrics on Cube Complexes*, ArXiv e-prints (2014).

[7] Arvin Moezzi, *The injective hull of hyperbolic groups*, Ph.D. thesis, ETH Zürich, 2010.

[8] Bożena Piątek, *On the gluing of hyperconvex metrics and diversities*, Ann. Univ. Paedagog. Crac. Stud. Math. **13** (2014), 65–76. MR 3231157